

# On a Voter model on $\mathbb{R}^d$ : Cluster growth in the Spatial $\Lambda$ -Fleming-Viot Process

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**Abstract:** The spatial  $\Lambda$ -Fleming-Viot (SAFV) process introduced in (Barton, Etheridge and Véber, 2010) can be seen as a direct extension of the Voter Model (Clifford and Sudbury, 1973); (Liggett, 1997). As such, it is an Interacting Particle System with configuration space  $\mathcal{M}^{\mathbb{R}^d}$ , where  $\mathcal{M}$  is the set of probability measures on some space  $K$ . Such processes are usually studied thanks to a dual process that describes the genealogy of a sample of particles. In this paper, we propose two main contributions in the analysis of the SAFV process. The first is the study of the growth of a cluster, and the surprising result is that with probability one, every bounded cluster stops growing in finite time. In particular, we discuss why the usual intuition is flawed. The second contribution is an original method for the proof, as the traditional (backward in time) duality methods fail. We develop a forward in time method that exploits a martingale property of the process. To make it feasible, we construct adequate objects that allow to handle the complex geometry of the problem. We are able to prove the result in any dimension  $d$ .

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## 1. Introduction

The spatial  $\Lambda$ -Fleming-Viot process (SAFV) is a model used to represent biological evolution on a continuum. It was first introduced in [7], and then studied in more details in [1], [2] and [3]. In this setting, given a set of genetic types  $K$ , a population living on  $\mathbb{R}^d$  is represented by a collection of probability measures on  $K$ . More precisely, the genetic composition at time  $t$  of the population at point  $x \in \mathbb{R}^d$  is given by a measure  $\rho_t(x, \cdot)$  on the type space  $K$ . The SAFV process is a direct spatial extension of the generalised Fleming-Viot processes presented in [6] and studied in [4]. But it can also be seen as an interacting particle system generalising the Voter Model [5, 8]. The configuration space for the Voter Model is  $\{0, 1\}^{\mathbb{Z}^d}$ , whereas for the SAFV process it is  $\mathcal{M}^{\mathbb{R}^d}$ , where  $\mathcal{M}$  is the set of probability measures on  $K$ . This generalisation of the configuration space is one of the elements that make the study of the SAFV process particu-

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larly challenging.

Our motivation in this article is the study of the fate of a new genetic type created by mutation at time 0. More precisely, we assume that there are only two types of individuals, *Blue* and *Red*, and that the new type, say *Red*, occupies a bounded set of  $\mathbb{R}^d$  at time 0. The question is how far this newly created type is going to spread.

Because we are working with two types only, the setting simplifies. We have  $\rho_t(x, \text{Blue}) = 1 - \rho_t(x, \text{Red})$ , so at time  $t$  it is enough to consider the collection of numbers  $\{\rho_t(y, \text{Red}), y \in \mathbb{R}^d\}$ . This is why we are going to represent the population at time  $t$  by the function

$$X_t : \mathbb{R}^d \mapsto [0, 1]$$

such that  $X_t(y) = \rho_t(y, \text{Red})$ . Working with a function instead of a collection of probability measures allows us to simplify the notation when manipulating the SAFV process.

### 1.1. The process

For every time  $t \geq 0$ , let  $X_t$  be a function from  $\mathbb{R}^d$  to  $[0, 1]$ . The quantity  $X_t(y)$  for  $y \in \mathbb{R}^d$  is the frequency of *Red* individuals at location  $y$ . The dynamics of  $X_t$  is the following. Consider a space-time Poisson point process  $\Pi$  on  $\mathbb{R}_+ \times \mathbb{R}^d \times [0, 1]$  with rate  $dt \otimes dc \otimes dv$ , and two constants  $0 \leq U < 1$  and  $R > 0$ . Then, for every point  $(t, c, v)$  of  $\Pi$ ,

- i) Draw a ball  $B(c, R)$  of radius  $R$  centred around  $c$ .
- ii) If  $X_{t-}(c) \geq v$ , then the parent is *Red*, and for every point  $y \in B(c, R)$ ,

$$X_t(y) = (1 - U)X_{t-}(y) + U.$$

- iii) If  $X_{t-}(c) < v$ , then the parent is *Blue*, and for every point  $y \in B(c, R)$ ,

$$X_t(y) = (1 - U)X_{t-}(y).$$

The steps i), ii) and iii) can be written in a single equation:

$$X_t(y) = X_{t-}(y) + U \mathbb{1}_{\{\|y-c\| \leq R\}} (\mathbb{1}_{\{v \leq X_{t-}(c)\}} - X_{t-}(y)), \quad y \in \mathbb{R}^d. \quad (1)$$

In biological terms, each point  $t, c, R$  of the Poisson Point process corresponds to a *reproduction event* taking place at time  $t$  in a ball  $B(c, R)$ . First, a parent is chosen at random at location  $c$ . The parent is *Red* with probability  $X_{t-}(c)$  and *Blue* with probability  $1 - X_{t-}(c)$ , and her offspring are going to have the same type as her. Second, competition for finite resources causes a proportion  $U$  of the population inside the ball of centre  $c$  and radius  $R$  to die. Finally,

the offspring of the parent replaces the proportion  $U$  of individuals who have died. Births and deaths take place simultaneously at time  $t$ . Figure 1 illustrates the births and deaths events taking place during a single transition at time  $t$  corresponding to the point  $(t, c, v)$  from the Poisson point process  $\Pi$ .

**Remark 1.1.** *We have chosen the parent to be at location  $c$ , which is a simplification of the model in [2], where the location of the parent was chosen uniformly on the ball. This does not change the model significantly, it just simplifies some calculations.*

The presentation of the process we just gave is simply an algorithm that describes the jumps of  $(X_t)_{t \geq 0}$ , but we need to construct it formally as a Markov process. The most natural way is to translate this algorithm into the infinitesimal generator  $\mathcal{L}$  of  $X(t)$ , which is defined by

$$\mathcal{L}I(f) := \lim_{t \rightarrow 0} \frac{\mathbb{E}[I(X_t) - I(X_0) \mid X_0 = f]}{t}, \quad (2)$$

where  $I$  is a test function, and  $f$  is the initial value of the process  $(X_t)_{t \geq 0}$ . We choose the test function  $I$  from the family  $I_n(\cdot; \psi)$  of functions of the form

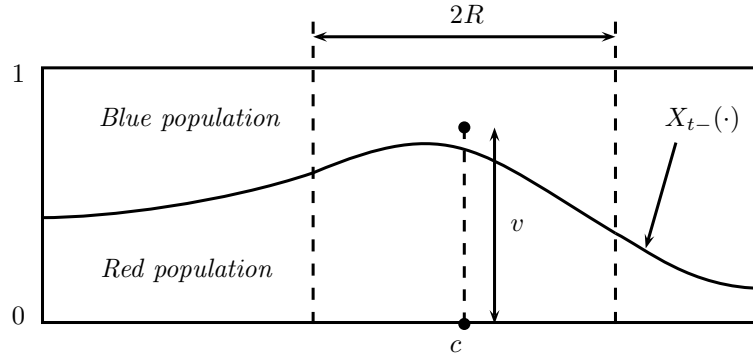
$$I_n(f; \psi) = \int_{(\mathbb{R}^d)^n} \psi(x_1, \dots, x_n) \prod_{i=1}^n f(x_i) dx_1 \dots dx_n, \quad (3)$$

where  $\psi$  is a function from  $(\mathbb{R}^d)^n$  to  $\mathbb{R}$  such that  $\int |\psi(x_1, \dots, x_n)| dx_1 \dots dx_n < \infty$ , and  $f$  is a function from  $(\mathbb{R}^d)^n$  to  $[0, 1]$  corresponding to  $X_0$ . The intuition behind this form is that the distribution of the function-valued process  $(X_t)_{t \geq 0}$  is described by the finite-dimensional dynamics at all locations  $x_1, \dots, x_n$ .

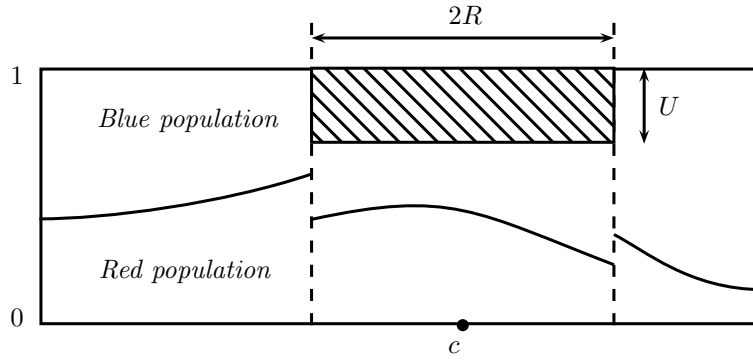
The generator  $\mathcal{L}$  of the process  $(X_t)_{t \geq 0}$  is given by

$$\begin{aligned} & \mathcal{L}I_n(f; \psi) \\ &= \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^n} \sum_{I \subset \{1, \dots, n\}} \left( \prod_{j \notin I} \mathbb{1}_{x_j \notin B(c, R)} f(x_j) \right) \times \left( \prod_{i \in I} \mathbb{1}_{x_i \in B(c, R)} \right) \\ & \quad \times \left[ f(c) \left( \prod_{i \in I} \left( (1 - U)f(x_i) + U \right) - \prod_{i \in I} f(x_i) \right) \right. \\ & \quad \left. + (1 - f(c)) \left( \prod_{i \in I} (1 - U)f(x_i) - \prod_{i \in I} f(x_i) \right) \right] \\ & \quad \psi(x_1, \dots, x_n) dx_1 \dots dx_n dc. \end{aligned} \quad (4)$$

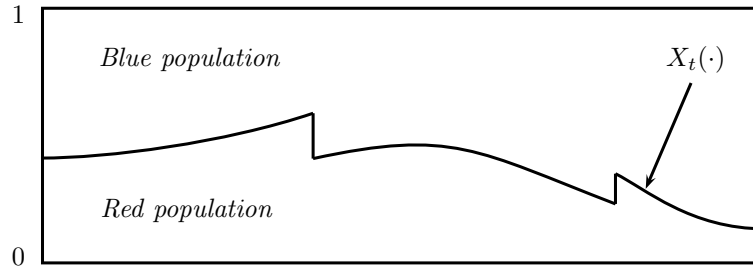
To understand this expression, we can think of  $(X_t)_{t \geq 0}$  as a jump process with possibly an infinite number of jumps at each instant. The transitions of the process are indexed by the points  $(t, c, v)$  of the Poisson Point process  $\Pi$  with



(a) Sampling of the parent



(b) Deaths



(c) Births

FIG 1. Schematic view in dimension  $d = 1$  of a Markov transition for the process  $X_t(\cdot)$  induced by the point  $(t, c, v) \in \Pi$ . (a) Because  $v > X_{t-}(c)$ , the parent is Blue. (b) Deaths shrink both populations within the ball  $B(c, R)$  by the same factor  $1 - U$ . (c) The offspring of the Blue parent replenish the population. All steps (a,b,c) take place instantaneously at time  $t$ .

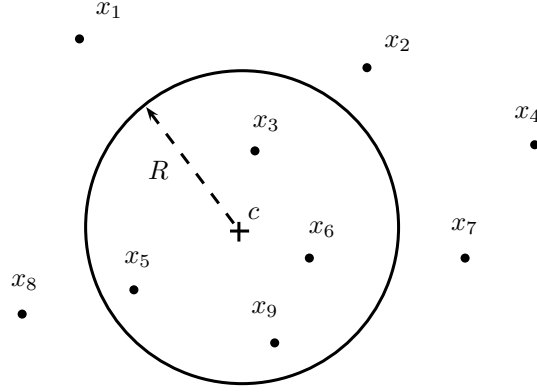


FIG 2. Visualisation of the unique set  $I$  that produces a nonzero term in the sum appearing in the definition (4) of the generator. In this figure,  $d = 2$ ,  $n = 9$ , and the set  $I = \{3, 5, 6, 9\}$  indexes the points  $x_i$  that belong to the ball of centre  $c$  and radius  $R$ .

intensity  $dt \otimes dc \otimes dv$ . Morally, we can use equation (1) and write the generator in the form

$$\mathcal{L}I_n(f; \psi) = \int_{\mathbb{R}^d} \int_{[0,1]} \left[ I_n(f_{(c,v)}; \psi) - I_n(f; \psi) \right] dv dc,$$

where

$$f_{(c,v)}(x) = f(x) + U \mathbb{1}_{\{\|x-c\| \leq R\}} \left( \mathbb{1}_{\{v \leq f(c)\}} - f(x) \right).$$

When we replace the test functions  $I_n$  with their expression, we obtain

$$\begin{aligned} \mathcal{L}I_n(f; \psi) = \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^n} \int_{[0,1]} & \left[ \prod_{i=1}^n f_{(c,v)}(x_i) - \prod_{i=1}^n f(x_i) \right] dv \\ & \psi(x_1, \dots, x_n) dx_1 \dots dx_n dc. \end{aligned}$$

To express the integral

$$\int_{[0,1]} \left[ \prod_{i=1}^n f_{(c,v)}(x_i) - \prod_{i=1}^n f(x_i) \right] dv,$$

we find the unique set  $I \subset \{1, \dots, n\}$  that verifies  $x_j \in B(c, R)$  if and only if  $j \in I$  (see figure 2). After careful computations, we obtain expression (4).

One needs to prove that there exists a Markov process  $(X_t)_{t \geq 0}$  that is defined by (4). A general proof for the existence of the SAFV process is given in [2] using duality. However, we do not need to use this result, because in our case we are able to construct directly the process using our forward in time method, see §5.3.

### 1.2. Main result

The bounded support *Red* population is competing against the unbounded support *Blue* population, so intuitively, we expect the *Red* population to become extinct. The real question is how far the *Red* population manages to spread before ultimately disappearing. This is why we are studying the dynamics of the support of the *Red* population.

Every time the individual sampled to be the parent is not *Red*, the proportion of the *Red* population decreases, which decreases the overall probability that the parent at the next event is going to be *Red*. The same reasoning applies to the *Blue* population. The only way for the support to grow is if the ball of centre  $c$  and radius  $R$  is not entirely contained in the support of  $X_t$ , while the parent sampled at  $c$  is *Red*. On the other hand, if we take  $U < 1$ , the support never shrinks, as once a point  $y \in \mathbb{R}^d$  is occupied with a frequency  $f(y)$ , its future frequencies will always be positive.

Given this schematic view of the dynamics of the process, one intuitively expects a behaviour similar to what we are going to call the *oil film spreading*: The proportion of the *Red* population would converge to zero at every point, but at the same time its support would grow forever and ultimately would occupy an infinite subset of  $\mathbb{R}^d$ . Stated more naturally, there seems to be no reason why the support would not grow to an infinitely large set.

However, the actual behaviour of the process is rather counterintuitive. Before stating our main result, we need to introduce some notation.

**Notation 1.2.** • For any given function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , we denote its support by  $\text{Supp}(f)$ .

- We define  $\mathcal{S}_c$  to be the set of Borel measurable functions  $f : \mathbb{R}^d \rightarrow [0, 1]$  with compact support. We endow  $\mathcal{S}_c$  with the  $L^\infty$  norm.
- Given a set  $A \subset \mathbb{R}^d$  and  $R > 0$ , we denote by  $A^R$  the  $R$ -expansion of  $A$ , that is the set defined by

$$A^R := \{x \in \mathbb{R}^d \text{ s.t. } \min_{y \in A} \|x - y\| \leq R\}.$$

**Theorem 1.3.** Let  $(X_t)_{t \geq 0}$  be a Markov process with generator (4). Suppose  $X_0 = f$  is deterministic and with bounded support. Then, there exists a random finite set  $B \subset \mathbb{R}^d$ , and an almost surely finite random time  $T$  such that

$$\forall t > T, \text{Supp}(X_t) = B \quad \text{a.s.} \quad (5)$$

Furthermore,

$$\sup_{z \in B} X_t(z) \rightarrow 0 \quad \text{a.s. as } t \rightarrow \infty \quad (6)$$

**Remark 1.4.** For the sake of clarity, we chose  $X_0$  to be deterministic. The result would still be true if  $\mathbb{E}[|\text{Supp}(X_0)|] < \infty$ , see Remark 4.12.

The proof of this result is the objective of this paper. It is fairly challenging because of the large dimension of the state space. Although the result is stated for the support of  $X_t$ , the actual object we need to keep track of is the whole function  $X_t$ . The natural approach consisting in approximating probabilities of trajectories that correspond to the event in question simply do not work, because such approximations waste too much information about the process. Our approach is to first summarise the structure of the process in a useful way. This is why we build adequate geometric tools that allow us to use a powerful martingale argument.

Fundamentally, the cause of the behaviour described in Theorem 1.3 lies in the discrete nature of the jumps inherent to  $\Lambda$ FV processes, rather than in the geometry of the SAFV process.

To see that, we consider the simpler process  $(\hat{Z}_t)_{t \geq 0}$  where there is no space, that is to say at every reproduction event, the parent is sampled with probability  $\hat{Z}_t$ , and a proportion  $U$  of the whole population is replaced by the offspring of the parent. In the notation of [4], the process  $(\hat{Z}_t)_{t \geq 0}$  is the  $\Lambda$ FV process on the state-space  $K = \{Red, Blue\}$  with  $\Lambda$ -measure  $\Lambda(du) = u^2 \delta_U(du)$ . It is a continuous time Markov Chain on  $[0, 1]$  with constant intensity, and the transitions of its embedded discrete time Markov Chain  $(Z_n)_{n \geq 0}$  are given by

$$\begin{cases} Z_{n+1} = (1 - U)Z_n + U \varepsilon_{n+1}, \\ \varepsilon_{n+1} | Z_n \sim B(Z_n), \end{cases}$$

where  $B(Z_n)$  is a Bernoulli distribution with parameter  $Z_n$ . It is straightforward to show that  $(Z_n)_{n \geq 0}$  is a nonnegative martingale, and therefore converges almost surely. As a consequence,  $\varepsilon_n$  converges almost surely to 0 or 1. This means that after some finite random time,  $\varepsilon_n$  will remain constant equal to 0 or 1. Almost surely in finite time, either the *Red* or the *Blue* population will be the only one to keep reproducing.

This remarkable feature is due to the fact that if the frequency  $Z_n$  is not sampled a few times in a row, it is going to decrease geometrically, and becomes rapidly too small to be sampled again. The same reasoning applies to the *Blue* population.

As we will prove, the same mechanism takes place in the spatial model, that is after some almost surely finite random time, the *Red* population will stop reproducing.

### 1.3. Proofs and outline

The martingale argument we demonstrated in the previous section seems to be the most promising approach. However, to be able to use such an argument, we need to find a way to filter out all the complex dependencies introduced by space, which is the main challenge in this work. We solved this problem by

introducing in §4.2 the geometrical object of *forbidden region* that allows to connect the martingale convergence to the sampling of the *Red* population.

The rest of this article is devoted to the proof of Theorem 1.3, as well as the construction of the process  $(X_{(t)})_{t \geq 0}$  defined by (4).

In Section 2, we introduce a discrete time Markov Chain  $(Y_n)_{n \geq 0}$  which is the discrete time equivalent of  $(X_{(t)})_{t \geq 0}$ . This chain is going to be used to construct  $(X_{(t)})_{t \geq 0}$  and to prove Theorem 1.3. We state in Proposition 2.5 the equivalent of Theorem 1.3 in discrete time.

Section 3 provides a toolbox that allows to handle easily the geometry of the model.

We prove in Section 4.3 the central Proposition 4.11, which states that in discrete time, the *Red* population defined by  $Y_n(\cdot)$  is sampled from only finitely many times. The proof relies on the fact that the total mass of the population, i.e. the integral of  $Y_n(\cdot)$  over  $\mathbb{R}_d$ , is a martingale that converges almost surely. In §4.2, we introduce the crucial concept of forbidden region, and use it to prove Proposition 4.11.

We gather all the results in Section 5. Proposition 4.11 allows both to use  $(Y_n)_{n \geq 0}$  to construct  $(X_{(t)})_{t \geq 0}$  as a non explosive continuous time Markov Chain, and to prove Theorem 1.3.

We finally conclude by discussing some extensions of this work.

## 2. A discrete time Markov chain

### 2.1. Construction

**Definition 2.1.** Consider  $R > 0$  and  $0 < U < 1$ . Let  $y$  be an  $\mathcal{S}_c$ -valued random variable. We construct simultaneously random sequences  $(C_n)_{n \geq 1}$  and  $(V_n)_{n \geq 1}$ , a filtration  $(\mathcal{P}_n)_{n \geq 0}$ , and an  $\mathcal{S}_c$ -valued Markov chain  $(Y_n)_{n \geq 0}$ , using the following recurrence:

$$\begin{cases} Y_0 = y, \\ \mathcal{P}_0 := \sigma(Y_0), \\ \Delta_0 = \text{Supp}(Y_0). \end{cases} \quad (7)$$

and for  $n \geq 0$ ,

- $\mathcal{P}_n := \sigma(C_1, \dots, C_n, V_1, \dots, V_n, Y_0, \dots, Y_n)$ ,
- conditionally on  $\mathcal{P}_n$ ,  $C_{n+1}$  is uniform on  $\Delta_n^R$ ,
- $V_{n+1}$  is distributed uniformly on  $[0, 1]$ , independently from  $\mathcal{P}_n$  and  $C_{n+1}$ ,
- $Y_{n+1}$  is given by the formula

$$Y_{n+1}(\cdot) = Y_n(\cdot) + U \delta_{B(C_{n+1}, R)}(\cdot) (\mathbf{1}_{\{V_{n+1} \leq Y_n(C_{n+1})\}} - Y_n(\cdot)) \quad (8)$$

- $\Delta_{n+1} = \text{Supp}(Y_{n+1})$ .



We introduce the notation  $\varepsilon_{n+1} := \mathbb{1}_{\{V_{n+1} \leq Y_n(C_{n+1})\}}$ . In particular, the trajectories of  $(\Delta_n)_{n \geq 0}$  are given by

$$\Delta_n = \Delta_0 \cup \bigcup_{\substack{1 \leq k \leq n, \\ \varepsilon_k = 1}} B(C_k, R). \quad (9)$$

Finally, we denote the natural filtration of  $(Y_n)_{n \geq 0}$  by  $\mathcal{G}_n := \sigma(Y_0, \dots, Y_n)$ .

**Remark 2.2.** We recall that  $\Delta_n^R$  is the  $R$ -expansion of the set  $\Delta_n$ .

At each reproduction event, the random variable  $C_{n+1}$  corresponds to the centre of the event. The parent is sampled uniformly at location  $C_{n+1}$  thanks to the random variable  $V_{n+1}$ , so that the parent is *Red* with probability  $Y_n(C_{n+1})$ , and *Blue* with probability  $1 - Y_n(C_{n+1})$ . The constants  $R$  and  $U$  are the radius of the event and the proportion of the population that is modified. The random variable  $\varepsilon_{n+1}$  indicates the types of the parent chosen.

**Notation 2.3.** If  $\varepsilon_{n+1} = 1$ , we say that the event is a positive sampling event, because the total red population increases, whereas when  $\varepsilon_{n+1} = 0$  we say that it is a negative sampling event.

Equation (9) shows that if the cluster  $\Delta_n$  is to increase, the minimum requirement is that there is a positive sampling.

**Remark 2.4.** Expression (9) is true because we assumed  $U < 1$ . In this case, the support and the range of the process coincide. Once a region is occupied by the Red population it remains occupied at every finite time. Therefore the cluster  $\Delta_n$  never shrinks. In the case where  $U = 1$  expression (9) would remain true if  $\Delta_n$  was defined to be the range of the process, that is  $\bigcup_{n \geq 0} \text{Supp}(Y_n)$ .

## 2.2. Result of the cluster convergence

The following result is the expression of our main result in the discrete time setting, with the temporary technical condition  $Y_0 = a \delta_{B(C_0, r_0)}$ . This condition is removed in Proposition 5.1 by allowing  $Y_0$  to be any deterministic function with bounded support.

**Proposition 2.5.** Suppose  $Y_0 = a \delta_{B(C_0, r_0)}$ , where  $a \in [0, 1]$ ,  $r_0 > 0$  and  $C_0 \in \mathbb{R}^d$ . Then, there exists an almost surely finite random time  $\kappa$  such that

$$\forall n > \kappa, \quad \varepsilon_n = 0. \quad (10)$$

Therefore, there exists an almost surely bounded random set  $B \in \mathbb{R}^d$  such that

$$\forall n > \kappa, \quad \Delta_n = B. \quad (11)$$

Most of the remainder of this paper is devoted to proving this Proposition. We first investigate the geometric properties of the model in the next section.

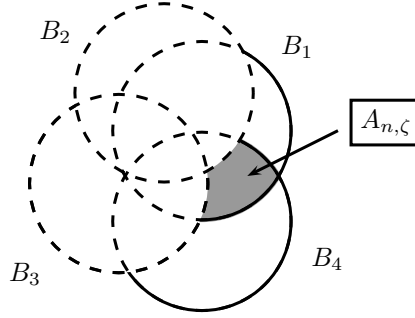


FIG 3. Structure of the level sets  $A_{n,\zeta}$  defined in (13), with  $n = 4$  and  $\zeta := \{1, 4\}$ . We use the notation  $B_j := B(C_j, R_j)$ .

### 3. Geometry

This section constructs all the tools that allow us to manage the geometry of the process.

**Remark 3.1.** From now on, unless specified otherwise, we suppose that  $Y_0 = a \delta_{B(C_0, r_0)}$ .

#### 3.1. $Y_n$ is piecewise constant

**Definition 3.2.** For notational convenience, we introduce the sequence  $(R_n)_{n \geq 0}$  such that

$$\begin{cases} R_0 = r_0, \\ R_n = R, \quad n \geq 1. \end{cases} \quad (12)$$

**Lemma 3.3.** For every  $n \geq 0$ , for every  $\zeta \subset \{0, \dots, n\}$ , consider the set  $A_{n,\zeta}$  defined by

$$A_{n,\zeta} := \left( \bigcap_{m \in \zeta} B(C_m, R_m) \right) \setminus \left( \bigcup_{\substack{j \leq n, \\ j \notin \zeta}} B(C_j, R_j) \right). \quad (13)$$

The function  $Y_n$  can be written as

$$Y_n = \sum_{\zeta \subset \{0, \dots, n\}} \alpha_{n,\zeta} \delta_{A_{n,\zeta}}, \quad (14)$$

where the sets  $A_{n,\zeta}$  are all disjoint for a given  $n$ , and  $\alpha_{n,\zeta} \geq 0$ .

**Remark 3.4.** By construction,  $\forall z \in A_{n,\zeta}$ , we have  $Y_n(z) = \alpha_{n,\zeta}$ .

*Proof.* We first introduce the shorter notation

$$B_j := B(C_j, R_j), \quad j \geq 0.$$

The fact that the sets  $A_{n,\zeta}$  are all disjoint for a given  $n$  is straightforward, so we just need to prove (14). But before that, we need to show that

$$\bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} = \bigcup_{i=1}^n B_i, \quad (15)$$

and we proceed by induction on  $n$ . The statement is true for  $n = 0$ . Suppose now that it is true for some given  $n \geq 0$ . Let  $\zeta' \subset \{0, \dots, n+1\}$ .

- If  $\zeta' = \{n+1\}$ , then  $A_{n+1,\zeta'} = B_{n+1} \setminus \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta}$ .
- If  $(n+1) \in \zeta'$ , and  $\zeta' \neq \{n+1\}$  then there exists  $\zeta \subset \{0, \dots, n\}$  such that

$$A_{n+1,\zeta'} = B_{n+1} \cap A_{n,\zeta}.$$

- If  $(n+1) \notin \zeta'$ , then there exists  $\zeta \subset \{0, \dots, n\}$  such that

$$A_{n+1,\zeta'} = A_{n,\zeta} \setminus B_{n+1}.$$

Therefore, we see that

$$\begin{aligned} \bigcup_{\zeta' \subset \{0, \dots, n+1\}} A_{n+1,\zeta'} &= \left( \bigcup_{\zeta \subset \{0, \dots, n\}} B_{n+1} \cap A_{n,\zeta} \right) \cup \left( \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \setminus B_{n+1} \right) \\ &\quad \cup \left( B_{n+1} \setminus \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \right) \\ &= \left( \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \right) \cup \left( B_{n+1} \setminus \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \right) \\ &= B_{n+1} \cup \left( \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \right), \end{aligned}$$

and the statement is proven using the inductive hypothesis.

We can return to the proof of expression (14). We need to show that for all  $n \geq 0$ ,  $\zeta \subset \{0, \dots, n\}$ ,

$$\left( x \notin \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n,\zeta} \right) \text{ implies } \left( Y_n(x) = 0 \right), \quad (16)$$

and

$$\left( x, y \in A_{n,\zeta} \right) \text{ implies } \left( Y_n(x) = Y_n(y) \right). \quad (17)$$

We prove (16) by induction on  $n$ , and we use (15). Statement (16) is satisfied for  $n = 0$  because  $Y_0 = a \delta_{B(C_0, r_0)}$ . Suppose now that it is true for some  $n \geq 0$ , and consider  $x \notin \cup_{i=1}^{n+1} B_i$ . In particular,  $x \notin B_{n+1}$ , and using the dynamics equation (8), we find that  $Y_{n+1}(x) = Y_n(x)$ . Because  $x \notin \cup_{i=1}^n B_i$ , we can use the inductive hypothesis, and we obtain that  $Y_n(x) = 0$ , which proves (16).

To prove (17), we also use induction. It is true for  $n = 0$ . Suppose (17) is satisfied for a given  $n \geq 0$ . Consider  $\zeta' \subset \{0, \dots, n+1\}$ , and take  $x, y \in A_{n+1, \zeta'}$ .

- If  $\zeta' = \{n+1\}$ , then  $A_{n+1, \zeta'} = B_{n+1} \setminus \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n, \zeta}$ , and in particular

$$x, y \notin \bigcup_{\zeta \subset \{0, \dots, n\}} A_{n, \zeta}.$$

We have  $\delta_{B_{n+1}}(x) = \delta_{B_{n+1}}(y) = 1$ , and using (16) we see that  $Y_n(x) = Y_n(y) = 0$ .

- If  $(n+1) \in \zeta'$  and  $\zeta' \neq \{n+1\}$ , there exists  $\zeta \subset \{0, \dots, n\}$  such that

$$A_{n+1, \zeta'} = B_{n+1} \cap A_{n, \zeta}.$$

Therefore  $\delta_{B_{n+1}}(x) = \delta_{B_{n+1}}(y) = 1$ , and using the inductive hypothesis, we have  $Y_n(x) = Y_n(y)$ .

- If  $(n+1) \notin \zeta'$ , then there exists  $\zeta \subset \{0, \dots, n\}$  such that

$$A_{n+1, \zeta'} = A_{n, \zeta} \setminus B_{n+1}.$$

In this case  $\delta_{B_{n+1}}(x) = \delta_{B_{n+1}}(y) = 0$ , and using the inductive hypothesis, we have  $Y_n(x) = Y_n(y)$ .

We can express  $Y_{n+1}$  using the dynamics equation (8), and we obtain:

$$\begin{cases} Y_{n+1}(x) = Y_n(x) + U \delta_{B_{n+1}}(x) (\varepsilon_{n+1} - Y_n(x)) \\ Y_{n+1}(y) = Y_n(y) + U \delta_{B_{n+1}}(y) (\varepsilon_{n+1} - Y_n(y)). \end{cases}$$

We have proved that for any choice of  $\zeta'$ , all the terms in the above equations are the same for  $x$  and  $y$ , therefore  $Y_{n+1}(x) = Y_{n+1}(y)$ , and we have proved (17).  $\square$

### 3.2. Variation of the local average

A central tool for the rest of the work is the average of the function  $Y_n$  on a ball of radius  $R$  and centre  $x \in \mathbb{R}^d$ . It is important that  $R$  is the radius of the reproduction event, as this is what links the martingale introduced in the next section to the geometry of the process (see Lemma 4.2). We introduce the following function:

**Definition 3.5.**

$$\Phi_n(x) := \int_{B(x, R)} Y_n(z) dz. \quad (18)$$

The main result of this section is the following.

**Proposition 3.6.** *For every  $x, y \in \mathbb{R}^d$ ,*

$$\Phi_n(y) - \Phi_n(x) \leq \|y - x\| S(R). \quad (19)$$

The rest of this section is devoted to proving this inequality. For this we need to introduce some auxiliary functions.

**Definition 3.7.** *Given  $x, y \in \mathbb{R}^d$ , for every  $n \geq 0$ , we define*

$$\begin{aligned} \Lambda_n^{x,y} : [0, \|y - x\|] &\longrightarrow [0, \infty) \\ t &\longmapsto \Lambda_n(t) := \Phi_n\left(x + t \frac{y - x}{\|y - x\|}\right). \end{aligned} \quad (20)$$

The key property for the proof of Proposition 3.6 is the following.

**Proposition 3.8.**  *$\Lambda_n^{x,y}$  is a continuous, piecewise differentiable function. Moreover, for every point  $t$  where  $\Lambda_n^{x,y}$  is differentiable, we have:*

$$\frac{d\Lambda_n^{x,y}}{dt}(t) \leq S(R). \quad (21)$$

*Proof.* We first prove that  $\Lambda_n^{x,y}$  is continuous and that there is at most a finite number  $J$  of points  $t_1 < \dots < t_J$  at which  $\Lambda_n^{x,y}$  is not differentiable. Thanks to equation (14) from Lemma 3.3, we see that  $\Phi_n$  is given by

$$\begin{aligned} \Phi_n(x) &= \int_{B(x,R)} \sum_{\zeta \subset \{0, \dots, n\}} \alpha_{n,\zeta} \delta_{A_{n,\zeta}}(z) dz \\ &= \sum_{\zeta \subset \{0, \dots, n\}} \alpha_{n,\zeta} |B(x, R) \cap A_{n,\zeta}|, \end{aligned}$$

therefore  $\Lambda_n^{x,y}(t)$  is given by

$$\Lambda_n^{x,y}(t) = \sum_{\zeta \subset \{0, \dots, n\}} \alpha_{n,\zeta} |B\left(x + t \frac{y - x}{\|y - x\|}, R\right) \cap A_{n,\zeta}|. \quad (22)$$

We simplify the notation by introducing  $B_j := B(C_j, R_j)$ ,  $j \geq 0$  and

$$x_t := x + t \frac{y - x}{\|y - x\|}.$$

The definition (13) of the sets  $A_{n,\zeta}$  and the inclusion-exclusion formula allow us to prove that for each  $\zeta \subset \{0, \dots, n\}$ , there exists a  $\beta_{n,\zeta} \in \mathbb{R}$  such that

$$\Lambda_n^{x,y}(t) = \sum_{\zeta \subset \{0, \dots, n\}} \beta_{n,\zeta} |B(x_t, R) \cap \left(\bigcap_{m \in \zeta} B_m\right)|.$$

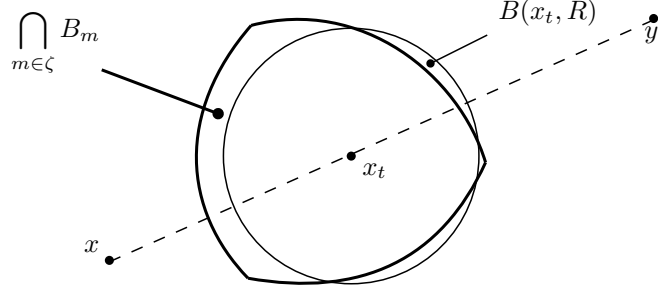


FIG 4. Visualisation of the function  $H_\zeta$  defined in (23). The points  $x$  and  $y$  are fixed, as well as the balls  $B_m$ ,  $m \in \zeta$ , with  $\zeta = \{1, 2, 3\}$  in this example. As the parameter  $t$  varies between 0 and  $\|y - x\|$ , the mobile point  $x_t$  moves along the segment joining  $x$  to  $y$ , with  $x_0 = x$  and  $x_{\|y-x\|} = y$ . For a given  $t$ ,  $H_\zeta(t)$  measures the volume of the intersection between the mobile ball  $B(x_t, R)$  and the fixed set  $\bigcap_{m \in \zeta} B_m$ . The function  $\Lambda_n^{x,y}$  defined in (20) is expressed as a finite linear combination of such functions  $H_\zeta$ , see (24)

**Remark 3.9.** The main change with expression (22) is that now we are working with intersections of balls, which are convex, whereas the sets  $A_{n,\zeta}$  are usually not. Also, we had the fact that  $\alpha_{n,\zeta}$  is the value of the function  $Y_n$  on the set  $A_{n,\zeta}$ , and such an interpretation is lost for  $\beta_{n,\zeta}$ .

If we introduce the function  $H_\zeta$  defined for each set  $\zeta \subset \{0, \dots, n\}$  by

$$\begin{aligned} H_\zeta : [0, \|y - x\|] &\longrightarrow \mathbb{R} \\ t &\longmapsto |B(x_t, R) \cap \bigcap_{m \in \zeta} B_m|, \end{aligned} \quad (23)$$

then we can simply rewrite the function  $\Lambda_n^{x,y}$  as

$$\Lambda_n^{x,y}(t) = \sum_{\zeta \subset \{0, \dots, n\}} \beta_{n,\zeta} H_\zeta(t). \quad (24)$$

The continuity of  $H_\zeta$  follows from the continuity of the function  $t \mapsto x_t$ , and this shows that  $\Lambda_n^{x,y}$  is continuous.

The set  $\bigcap_{m \in \zeta} B_m$  is convex, therefore there exist  $t_1, t_2$  such that

$$H_\zeta(t) > 0 \Leftrightarrow t_1 < t < t_2. \quad (25)$$

Consider  $t$  such that  $t_1 < t < t_2$ . Thanks to (25), this means we can choose a point  $z$  belonging to the interior of  $B(x_t, R) \cap \bigcap_{m \in \zeta} B_m$ . Because the set  $B(x_t, R) \cap \bigcap_{m \in \zeta} B_m$  is convex, we can express its volume in  $d$ -dimensional spherical coordinates with  $z$  as the new origin. Given angular coordinates  $\phi := (\phi_1, \dots, \phi_{d-1})$ , we denote by  $p_\phi(t)$  the unique point of the boundary of  $B(x_t, R) \cap$

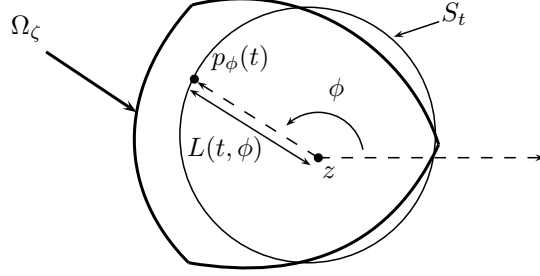


FIG 5. Illustration of  $p_\phi(t)$  in dimension 2. The point  $z$  is chosen arbitrarily inside the intersection of  $\bigcap_{m \in \zeta} B_m$  and  $B(x_t, R)$ , and is taken to be the new origin. The boundaries of  $\bigcap_{m \in \zeta} B_m$  and  $B(x_t, R)$  are denoted respectively by  $\Omega_\zeta$  and  $S_t$ . The angle  $\phi$  is defined in the local polar coordinate system, and for a given angle  $\phi$  the point  $p_\phi(t)$  is the projection of  $z$  along the angle  $\phi$  on the boundary of  $\bigcap_{m \in \zeta} B_m \cap B(x_t, R)$ . For some values of  $\phi$  it belongs to the mobile sphere  $S_t$ , and for other values it belongs to the static surface  $\Omega_\zeta$ .

$\bigcap_{m \in \zeta} B_m$  with angular coordinates  $\phi$ , and  $L(t, \phi)$  the distance between  $z$  and  $p_\phi(t)$ . We have:

$$H_\zeta(t) = \int_0^\pi \dots \int_0^\pi \int_0^{2\pi} \int_0^{L(t, \phi)} r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) dr d\phi_{d-1} \dots d\phi_1.$$

We denote by  $\Omega_\zeta$  the boundary of  $\bigcap_{m \in \zeta} B_m$ , and by  $S_t$  the sphere of centre  $x_t$  and radius  $R$ . We can find a partition  $\Xi_1, \dots, \Xi_K$  of the space

$$[0, \pi]^{d-2} \times [0, 2\pi]$$

such that

$$\begin{cases} \forall \phi \in \Xi_j, p_\phi(t) \in \Omega_\zeta, \\ \text{or} \\ \forall \phi \in \Xi_j, p_\phi(t) \in S_t. \end{cases}$$

Therefore we can write  $H_\zeta(t)$  as

$$H_\zeta(t) = \sum_{j=1}^K \int_{\Xi_j} \int_0^{L(t, \phi)} r^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) dr d\phi.$$

Thanks to this representation, we see that a sufficient condition for  $H_\zeta$  to be differentiable at  $t$ ,  $t_1 < t < t_2$ , is that for every  $\phi$  in the interior of every  $\Xi_j$ , the function  $t \mapsto L(t, \phi)$  is differentiable at  $t$ . In this case, the derivative is given by

$$\frac{dH_\zeta(t)}{dt} = \sum_{j=1}^K \int_{\Xi_j} \frac{\partial L}{\partial t}(t, \phi) L(t, \phi)^{d-1} \sin^{d-2}(\phi_1) \sin^{d-3}(\phi_2) \dots \sin(\phi_{d-2}) d\phi.$$

We focus now on the differentiability of  $t \mapsto L(t, \phi)$ , where  $\phi$  belongs to the interior of  $\Xi_j$  for some  $j$ .

Suppose first that  $p_\phi(t) \in \Omega_\zeta$ . Because the function  $t \mapsto x_t$  is continuous, and because  $z$  is a fixed point, there exists  $h > 0$  such that for every  $u \in (t-h, t+h)$ ,  $p_\phi(u) \in \Omega_\zeta$ . Therefore, there exists  $h > 0$  such that for every  $u \in (t-h, t+h)$ ,  $L(u, \phi) = L(t, \phi)$ , and  $t \mapsto L(t, \phi)$  is differentiable at  $t$ .

In the case where  $p_\phi(t) \in S_t$ , by the same continuity argument, we obtain that for every  $u \in (t-h, t+h)$ ,  $p_\phi(u) \in S_u$ . Because the distance between  $z$  and the projection of  $z$  on  $S_t$  along the angle  $\phi$  is differentiable, we conclude that  $t \mapsto L(t, \phi)$  is also differentiable in this case.

We have proved that for all  $t \neq t_1, t_2$ ,  $H_\zeta$  is differentiable at  $t$ . Given that

$$\Lambda_n^{x,y}(t) = \sum_{\zeta \in \{0, \dots, n\}} \beta_{n,\zeta} H_\zeta(t),$$

we conclude that there is at most a finite number of points at which  $\Lambda_n^{x,y}$  is not differentiable, and this proves the first part of the Proposition.

We need now to show the upper bound (21) for the derivative. Suppose  $\Lambda_n^{x,y}$  is differentiable at  $t$ . By definition,

$$\Lambda_n^{x,y}(t) = \int_{B(x_t, R)} Y_n(z) dz,$$

therefore

$$\Lambda_n^{x,y}(t+h) - \Lambda_n^{x,y}(t) = \int_{B(x_{t+h}, R)} Y_n(z) dz - \int_{B(x_t, R)} Y_n(z) dz.$$

By construction, for all  $h \geq 0$ ,  $B(x_{t+h}, R) \subset B(x_t, R+h)$ , which implies that

$$\begin{aligned} \Lambda_n^{x,y}(t+h) - \Lambda_n^{x,y}(t) &\leq \int_{B(x_t, R+h)} Y_n(z) dz - \int_{B(x_t, R)} Y_n(z) dz \\ &\leq \int_{B(x_t, R+h) \setminus B(x_t, R)} Y_n(z) dz \\ &\leq |B(x_t, R+h)| - |B(x_t, R)|. \end{aligned}$$

Dividing by  $h$  and taking the limit as  $h \rightarrow 0$ , we obtain

$$\frac{d\Lambda_n^{x,y}}{dt}(t) \leq \frac{dV(R)}{dR} = S(R), \quad (26)$$

where  $V(R)$  is the volume of a ball of radius  $R$ , and  $S(R)$  its surface area.  $\square$



*Proof of Proposition 3.6.* We saw in the previous proof that there is only a finite number of points  $t_1, \dots, t_J$  at which  $\Lambda_n^{x,y}$  is not differentiable. By continuity of  $\Lambda_n^{x,y}$ , and using Proposition 3.8,

$$\begin{aligned}
& \Phi_n(y) - \Phi_n(x) \\
&= \Lambda_n^{x,y}(\|y - x\|) - \Lambda_n^{x,y}(0) \\
&= \Lambda_n^{x,y}(\|y - x\|) - \Lambda_n^{x,y}(t_J) + \sum_{j=1}^{J-1} \Lambda_n^{x,y}(t_{j+1}) - \Lambda_n^{x,y}(t_j) \\
&\quad + \Lambda_n^{x,y}(t_1) - \Lambda_n^{x,y}(0) \\
&\leq S(R)(\|y - x\| - t_J) + \sum_{j=1}^{J-1} S(R)(t_{j+1} - t_j) \\
&\quad + S(R)(t_1 - 0) \\
&\leq \|y - x\| S(R). \square
\end{aligned}$$

#### 4. Probability

##### 4.1. A martingale argument

**Definition 4.1.** We denote by  $M_n$  the total mass of  $Y_n$ , that is

$$M_n = \int_{\mathbb{R}^d} Y_n(z) dz.$$

**Lemma 4.2.** The change of the total mass  $M_{n+1} - M_n$  is given by

$$M_{n+1} - M_n = U(\varepsilon_{n+1}V(R) - \Phi_n(C_{n+1})) \quad (27)$$

*Proof.* Using (8),

$$\begin{aligned}
M_{n+1} - M_n &= \int_{\mathbb{R}^d} (Y_{n+1}(z) - Y_n(z)) dz \\
&= \int_{\mathbb{R}^d} U \delta_{B(C_{n+1}, R)}(z) (\varepsilon_{n+1} - Y_n(z)) dz \\
&= U \int_{B(C_{n+1}, R)} (\varepsilon_{n+1} - Y_n(z)) dz \\
&= U(\varepsilon_{n+1}V(R) - \Phi_n(C_{n+1})) \square
\end{aligned}$$

**Proposition 4.3.**  $(M_n)_{n \geq 0}$  is a discrete time nonnegative  $(\mathcal{G}_n)_{n \geq 0}$  martingale.

*Proof.* Thanks to Lemma 4.2 and Definition 2.1, we can calculate explicitly  $\mathbb{E}[M_{n+1} - M_n \mid \mathcal{G}_n]$  as follows:

$$\begin{aligned}
& \mathbb{E}[M_{n+1} - M_n | \mathcal{G}_n] \\
&= \mathbb{E} \left[ U \mathbb{1}_{\{V_{n+1} \leq Y_n(C_{n+1})\}} \int_{B(C_{n+1}, R)} dz - U \int_{B(C_{n+1}, R)} Y_n(z) dz \mid \mathcal{G}_n \right] \\
&= U \int_{\mathbb{R}^d} \int_{[0,1]} \left[ \mathbb{1}_{\{v \leq Y_n(c)\}} \int_{B(c, R)} dz - \int_{B(c, R)} Y_n(z) dz \right] dv \frac{\mathbb{1}_{c \in \Delta_n^R}}{|\Delta_n^R|} dc \\
&= U \int_{\mathbb{R}^d} \left[ Y_n(c) \int_{B(c, R)} dz - \int_{B(c, R)} Y_n(z) dz \right] \frac{\mathbb{1}_{c \in \Delta_n^R}}{|\Delta_n^R|} dc \\
&= \frac{U}{|\Delta_n^R|} \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{z \in B(c, R)} \mathbb{1}_{c \in \Delta_n^R} Y_n(c) dz dc \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{z \in B(c, R)} \mathbb{1}_{c \in \Delta_n^R} Y_n(z) dz dc \right].
\end{aligned}$$

In particular, for every  $f \in \mathcal{S}_c$ ,

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{z \in B(c, R)\}} \mathbb{1}_{\{c \in \text{Supp}(f)^R\}} f(c) dz dc \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{c \in B(z, R)\}} f(c) dz dc \quad \text{since } c \in B(z, R) \Leftrightarrow z \in B(c, R) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{z \in B(c, R)\}} f(z) dc dz, \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{\{z \in B(c, R)\}} \mathbb{1}_{\{c \in \text{Supp}(f)^R\}} f(z) dc dz,
\end{aligned}$$

so  $\mathbb{E}[M_{n+1} - M_n | \mathcal{G}_n] = 0$ , which shows that  $(M_n)_{n \geq 0}$  is a martingale.  $\square$

**Definition 4.4.** Let  $\alpha > 0$  be a real number such that  $0 < \alpha < UV(R)/2$ . We then define

$$\tau_\alpha := \inf\{p \geq 0 : \forall n \geq p, \quad |M_{n+1} - M_n| < \alpha\}. \quad (28)$$

In particular,  $\tau_\alpha$  is not a stopping time, but this is not going to be an issue in what follows.

**Proposition 4.5.** The random time  $\tau_\alpha$  is a.s. finite, and  $\forall n > \tau_\alpha$ ,

$$\begin{cases} \Phi_n(C_{n+1}) < \frac{\alpha}{U} & \text{if } \varepsilon_{n+1} = 0, \\ \Phi_n(C_{n+1}) > V(R) - \frac{\alpha}{U} & \text{if } \varepsilon_{n+1} = 1. \end{cases} \quad (29)$$

*Proof.* We know that  $M_n$  is a nonnegative martingale, so it converges almost surely when  $n \rightarrow \infty$ . Therefore  $\tau_\alpha$  is almost surely finite, and by definition,  $\forall n > \tau_\alpha$ ,  $|M_{n+1} - M_n| < \alpha$ . Using Lemma 4.2, we observe that

$$|M_{n+1} - M_n| = \begin{cases} U \Phi_n(C_{n+1}) & \text{if } \varepsilon_{n+1} = 0, \\ U (V(R) - \Phi_n(C_{n+1})) & \text{if } \varepsilon_{n+1} = 1, \end{cases}$$

which concludes the proof.  $\square$

#### 4.2. Forbidden region

The concept of a forbidden region will allow us to treat probabilistically the geometric properties established in §3.

**Lemma 4.6.** *For  $n \geq 0$ ,*

$$\begin{aligned} & \{n \geq \tau_\alpha\} \cap \{\varepsilon_k = 1 \text{ infinitely often}\} \\ & \subset \{n \geq \tau_\alpha\} \cap \bigcap_{j=n}^{\infty} \{\sup \Phi_j > V(R) - \alpha/U\} \end{aligned} \quad (30)$$

*Proof.* Take  $j \geq \tau_\alpha$  such that  $\sup \Phi_j \leq V(R) - \alpha/U$ . Using Proposition 4.5, this implies that  $\varepsilon_{j+1} = 0$ . In particular,  $\sup \Phi_{j+1} \leq \sup \Phi_j \leq V(R) - \alpha/U$ . By induction, we just showed that

$$\begin{cases} j \geq \tau_\alpha \\ \sup \Phi_j \leq V(R) - \alpha/U \end{cases} \implies \begin{cases} j \geq \tau_\alpha \\ \forall k \geq j, \varepsilon_k = 0. \end{cases}$$

The contrapositive of this implication allows to conclude this proof.  $\square$

**Definition 4.7.** *We define the forbidden region  $F_n$  to be*

$$F_n := \{x \in \mathbb{R}^d : \frac{\alpha}{U} \leq \Phi_n(x) \leq V(R) - \frac{\alpha}{U}\}. \quad (31)$$

*We also introduce the quantity*

$$\psi := V \left( \frac{V(R) - 2\alpha/U}{S(R)} \right). \quad (32)$$

The reason for the name *forbidden region* is motivated by the following lemma, which tells us that after the time  $\tau_\alpha$ , if the local averages are always too high, then the points  $C_{n+1}$  are forbidden from falling in the region  $F_n$ . Furthermore, this lemma provides a lower bound on the volume of  $F_n$ .

**Lemma 4.8.**

$$\begin{aligned} & \{j \geq \tau_\alpha\} \cap \{\sup \Phi_j > V(R) - \alpha/U\} \\ & \subset \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \end{aligned} \quad (33)$$

*Proof.* An immediate consequence of Proposition 4.5 is

$$j \geq \tau_\alpha \Rightarrow C_{j+1} \notin F_j.$$

More work is required to obtain the lower bound for the volume of the forbidden region. We first define  $P_j := \{x \in \mathbb{R}^d : \Phi_j(x) \geq V(R) - \alpha/U\}$ . If we assume  $\sup \Phi_j > V(R) - \alpha/U$ , then  $P_j$  is nonempty. We can then take a point  $y \in P_j$ .

The function  $\Phi_j$  is continuous, and  $\Delta_j$  is finite, so the region  $N_j := \{x \in \mathbb{R}^d : \Phi_j(x) \leq \alpha/U\}$  is infinite. Indeed, for every  $x$  at a distance from  $\Delta_j$  larger than  $R$ ,  $\Phi_j(x) = 0$ . In particular, for a large enough positive number  $\bar{R}$ , we can consider  $\Gamma$  the sphere of radius  $\bar{R}$  and centre  $y$ , and the ball  $B(y, \bar{R})$  such that

$$\begin{cases} \Delta_j \subset B(y, \bar{R}), \\ \Gamma \subset N_j. \end{cases} \quad (34)$$

For  $x, y \in \mathbb{R}^d$ , we denote by  $[x, y]$  the line-segment between  $x$  and  $y$ . We need the following lemma:

**Lemma 4.9.** *The point  $y \in P_j$  being fixed, for every point  $x \in \Gamma$ , we can find two points  $x_0, y_0$  such that:*

$$\begin{cases} [x_0, y_0] \subset F_j, \\ [x_0, y_0] \subset [x, y], \\ \|y_0 - x_0\| \geq \frac{V(R) - 2\alpha/U}{S(R)}. \end{cases} \quad (35)$$

By integrating the result of Lemma 4.9 over all the points  $x \in \Gamma$ , we find that the volume of  $F_j$  is larger than the volume of a ball of radius  $(V(R) - 2\alpha/U)/S(R)$ , hence the result.  $\square$

*Proof of Lemma 4.9.* The function  $\Lambda_j^{x,y}$  defined in (20) is continuous, with  $\Lambda_j^{x,y}(0) = 0$  and  $\Lambda_j^{x,y}(\|y - x\|) \geq V(R) - \alpha/U$ , so there are two points  $t_1, t_2 \in [0, \|y - x\|]$  such that  $\Lambda_j^{x,y}([t_1, t_2]) = [\alpha/U, V(R) - \alpha/U]$ . By application of the continuous function  $t \rightarrow x + t(y - x)/(\|y - x\|)$ , this means that there are two points  $x_0, y_0 \in \mathbb{R}^d$  such that

$$\begin{aligned} - \Phi_j(x_0) &= \alpha/U, \\ - \Phi_j(y_0) &= V(R) - \alpha/U, \\ - \forall z \in (x_0, y_0), \Phi_j(z) &\in (\alpha/U, V(R) - \alpha/U). \end{aligned}$$

The last statement is just the fact that  $(x_0, y_0) \subset F_j$ . By using Corollary 3.6, we find that

$$\begin{aligned} \|y_0 - x_0\| S(R) &\geq \Phi_j(y_0) - \Phi_j(x_0) \\ &\geq V(R) - 2\alpha/U. \end{aligned}$$

$\square$

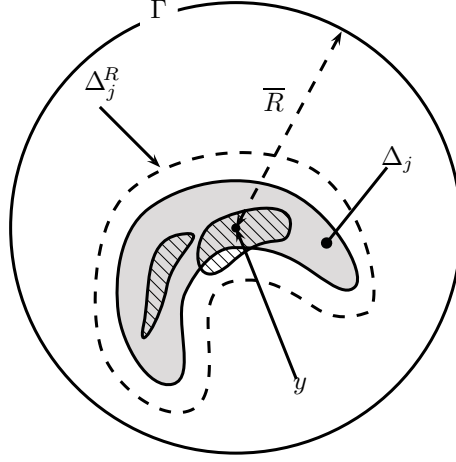


FIG 6. Illustration of the construction (34). The grey area  $\Delta_j$  is the support of  $Y_j$ , and the hashed area is the region  $P_j$ . We choose arbitrarily  $y \in P_j$ . The dashed line is the boundary of  $\Delta_j^R$ , the set of points at distance at most  $R$  from  $\Delta_j$ . For all  $x \notin \Delta_j^R$ ,  $\phi_j(x) = 0$ , therefore for large enough  $\bar{R}$ , the ball  $B(y, \bar{R})$  and its boundary  $\Gamma$  satisfy (34).

We reach now the main point of this section, which is an upper bound for the probability that infinitely many positive sampling events take place.

**Proposition 4.10.**

$$\begin{aligned} & \mathbb{P}(\varepsilon_k = 1 \text{ infinitely often}) \\ & \leq \sum_{l=0}^{\infty} \mathbb{P}\left(\bigcap_{j=l}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right) \end{aligned} \quad (36)$$

*Proof.* As  $\tau_\alpha < \infty$  a.s.,

$$\begin{aligned} & \mathbb{P}(\varepsilon_k = 1 \text{ i.o.}) \\ & = \mathbb{P}(\{\tau_\alpha < \infty\} \cap \{\varepsilon_k = 1 \text{ i.o.}\}) \\ & = \sum_{n=0}^{\infty} \mathbb{P}(\{\tau_\alpha = n\} \cap \{\varepsilon_k = 1 \text{ i.o.}\}) \\ & \leq \sum_{n=0}^{\infty} \mathbb{P}(\{n \geq \tau_\alpha\} \cap \{\varepsilon_k = 1 \text{ i.o.}\}) \end{aligned} \quad (37)$$

We can write  $\{n \geq \tau_\alpha\} = \bigcap_{j=n}^{\infty} \{j \geq \tau_\alpha\}$  Using this in (33), we obtain

$$\begin{aligned} & \{n \geq \tau_\alpha\} \cap \bigcap_{j=n}^{\infty} \{\sup \Phi_j > V(R) - \alpha/U\} \\ & \subset \bigcap_{j=n}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}. \end{aligned}$$

Combining this with (30), we have

$$\begin{aligned} & \{n \geq \tau_\alpha\} \cap \{\varepsilon_k = 1 \text{ i.o.}\} \\ & \subset \bigcap_{j=n}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}, \end{aligned} \tag{38}$$

and the result follows.  $\square$

#### 4.3. Finitely many positive sampling events

**Proposition 4.11.**

$$\mathbb{P}(\varepsilon_k = 1 \text{ infinitely often}) = 0.$$

*Proof.* Using Proposition 4.10, we simply need to prove that for every  $l \geq 0$ ,

$$\mathbb{P}\left(\bigcap_{j=l}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right) = 0.$$

By monotone convergence, we have

$$\mathbb{P}\left(\bigcap_{j=l}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right).$$

We are going to work in the slightly more general setting where  $Y_0$ , and therefore  $\Delta_0$ , are allowed to be random. This is easily dealt with, because we begin by conditioning on  $\Delta_0$ :

$$\mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right) = \mathbb{E}\left[\mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \mid \Delta_0\right)\right].$$

We then condition on all but the last reproduction events:

$$\begin{aligned} & \mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \mid \Delta_0\right) \\ &= \mathbb{E}\left[\mathbb{1}\{C_{l+1} \notin F_l, |F_l| \geq \psi\} \cdots \mathbb{1}\{C_n \notin F_{n-1}, |F_{n-1}| \geq \psi\} \right. \\ & \quad \left. \mathbb{P}\left(C_{n+1} \notin F_n, |F_n| \geq \psi \mid \Delta_0, F_l, C_{l+1}, \dots, F_{n-1}, C_n\right) \mid \Delta_0\right]. \end{aligned} \tag{39}$$

We can calculate the last term by conditioning on  $F_n$  and  $\Delta_n$ :

$$\begin{aligned}
& \mathbb{P}(C_{n+1} \notin F_n, |F_n| \geq \psi \mid \Delta_0, F_l, C_{l+1}, \dots, F_{n-1}, C_n, F_n, \Delta_n) \\
&= \mathbb{1}_{|F_n| \geq \psi} \mathbb{P}(C_{n+1} \notin F_n \mid F_n, \Delta_n) \\
&= \mathbb{1}_{|F_n| \geq \psi} \left(1 - \frac{|F_n|}{|\Delta_n^R|}\right) \\
&\leq 1 - \frac{\psi}{|\Delta_n^R|} \\
&\leq 1 - \frac{\psi}{|\Delta_0^R| + nV(2R)}. \tag{40}
\end{aligned}$$

The second and third equalities come from the fact that conditionally on  $\Delta_n$ ,  $C_{n+1}$  is sampled uniformly from  $\Delta_n^R$ , independently of the past. The last inequality comes from (9):

$$\Delta_n = \Delta_0 \cup \bigcup_{\substack{1 \leq k \leq n, \\ \varepsilon_k = 1}} B(C_k, R).$$

In particular, it implies that

$$\begin{aligned}
|\Delta_n^R| &\leq |\Delta_0^R| + |B(C_1, R)^R| + \dots + |B(C_n, R)^R| \\
&\leq |\Delta_0^R| + nV(2R).
\end{aligned}$$

Putting inequality (40) into (39), we obtain the following upper bound:

$$\begin{aligned}
& \mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \mid \Delta_0\right) \\
&\leq \left(1 - \frac{\psi}{|\Delta_0^R| + nV(2R)}\right) \mathbb{P}\left(\bigcap_{j=l}^{n-1} \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \mid \Delta_0\right). \tag{41}
\end{aligned}$$

Inequality (41) provides a recurrence relation, which we can solve immediately to obtain

$$\mathbb{P}\left(\bigcap_{j=l}^n \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \mid \Delta_0\right) \leq \prod_{j=l}^n \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right).$$

Taking expectation and then the limit as  $n \rightarrow \infty$ , we obtain

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{j=l}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\}\right) &\leq \lim_{n \rightarrow \infty} \mathbb{E}\left[\prod_{j=l}^n \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right)\right] \\
&\leq \mathbb{E}\left[\prod_{j=l}^{\infty} \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right)\right].
\end{aligned}$$

We rewrite the infinite random product using logarithms:

$$\begin{aligned} & \prod_{j=l}^{\infty} \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right) \\ &= \exp \left( \sum_{j=l}^{\infty} \log \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right) \right). \end{aligned}$$

After observing that

$$\log \left(1 - \frac{\psi}{|\Delta_0^R| + jV(2R)}\right) \underset{j \rightarrow \infty}{\overset{a.s.}{\sim}} \frac{-\psi/V(2R)}{j},$$

we conclude that the infinite product is almost surely equal to 0. Because we chose  $Y_0$  to be deterministic, we conclude that

$$\mathbb{P} \left( \bigcap_{j=l}^{\infty} \{C_{j+1} \notin F_j, |F_j| \geq \psi\} \right) = 0.$$

□

**Remark 4.12.** *In the case where we take  $Y_0$  to be random, a sufficient condition for the expectation of the infinite product to also be equal to 0 is simply  $\mathbb{E}(|\Delta_0|) < \infty$ , that is the volume of the initial support has a finite expectation.*

## 5. Proof of the theorems

### 5.1. Proof of Proposition 2.5

We proved in Proposition 4.11 that with probability one, there are only finitely many sampling events. This means that there exists an almost surely finite random time  $\kappa$  such that

$$\forall n > \kappa, \quad \varepsilon_n = 0. \quad (42)$$

We recall the dynamics of the cluster  $\Delta_n$  described by (9):

$$\Delta_n = \Delta_0 \cup \bigcup_{\substack{1 \leq k \leq n, \\ \varepsilon_k = 1}} B(C_k, R).$$

Therefore, if we define  $B := \Delta_\kappa$ , we have

$$\forall n > \kappa, \quad \Delta_n = B, \quad (43)$$

and the proof of Proposition 2.5 is complete.

We can now generalise Proposition 2.5 by removing the technical condition on the starting point and allowing  $Y_0$  to be any function in  $\mathcal{S}_c$ .



**Proposition 5.1.** *Suppose  $Y_0 = f \in \mathcal{S}_c$ . Then, there exists an almost surely finite random time  $\kappa$  such that*

$$\forall n > \kappa, \quad \varepsilon_n = 0. \quad (44)$$

*Therefore, there exists an almost surely bounded random set  $B \in \mathbb{R}^d$  such that*

$$\forall n > \kappa, \quad \Delta_n = B. \quad (45)$$

*Proof.* We proceed by coupling  $Y$  with a Markov chain  $\tilde{Y}$  with the same transition probabilities, but started from  $\tilde{Y}_0 = \delta_{B(C_0, r_0)}$  such that  $Y_0 \leq \tilde{Y}_0$ . We denote the initial conditions by  $\tilde{Y}_0 = \tilde{f}$  and  $Y_0 = f$ . We first build  $\tilde{Y}$  as described in Definition 2.1. We then use the sequences  $(\tilde{C}_n)_{n \geq 1}$  and  $(\tilde{V}_n)_{n \geq 1}$  that we used to construct  $\tilde{Y}$  in the following way. First consider the random sequence  $Y'$  defined by  $Y'_0 = f$ , and for  $n \geq 0$ ,

$$Y'_{n+1} = Y'_n + U \delta_{B(\tilde{C}_{n+1}, R)} (\mathbb{1}_{\{\tilde{V}_{n+1} \leq Y'_n(\tilde{C}_{n+1})\}} - Y'_n).$$

We can prove by induction that

$$\forall n \geq 0, Y'_n \leq \tilde{Y}_n. \quad (46)$$

It is of course true at  $n = 0$ , and then we just observe that if  $Y'_n \leq \tilde{Y}_n$ , then

$$\begin{aligned} & \tilde{Y}_{n+1} - Y'_{n+1} \\ &= (1 - U \delta_{B(\tilde{C}_{n+1}, R)}) (\tilde{Y}_n - Y'_n) \\ & \quad + U \delta_{B(\tilde{C}_{n+1}, R)} (\mathbb{1}_{\{\tilde{V}_{n+1} \leq \tilde{Y}_n(\tilde{C}_{n+1})\}} - \mathbb{1}_{\{\tilde{V}_{n+1} \leq Y'_n(\tilde{C}_{n+1})\}}) \\ & \geq 0. \end{aligned}$$

We denote by  $\tilde{\Delta}$  and  $\Delta'$  the respective sequences of supports, and in particular we have proved that

$$\forall n \geq 0, \Delta'_n \subset \tilde{\Delta}_n. \quad (47)$$

We define the sequence of  $(\sigma(\tilde{\mathcal{P}}_n, f))_{n \geq 0}$ -stopping times  $(J_n)_{n \geq 0}$  by setting

$$\begin{cases} J_0 = 0, \\ J_{n+1} = \inf\{k > J_n : \tilde{C}_k \in (\Delta'_{k-1})^R\}. \end{cases} \quad (48)$$

We now construct  $(Y_n)_{n \geq 0}$ ,  $(C_n)_{n \geq 1}$  and  $(V_n)_{n \geq 1}$  by taking

$$\begin{cases} Y_n := Y'_{J_n}, & n \geq 0 \\ C_n := \tilde{C}_{J_n}, & n \geq 1 \\ V_n := \tilde{V}_{J_n}, & n \geq 1. \end{cases}$$

We denote by  $\Delta_n$  the support of  $Y_n$ , and we define the filtration  $(\mathcal{P}_n)_{n \geq 0}$  to be

$$\begin{cases} \mathcal{P}_0 := \sigma(Y_0), \\ \mathcal{P}_n := \sigma(C_1, \dots, C_n, V_1, \dots, V_n, Y_0, \dots, Y_n), \end{cases}$$

By construction, conditionally on  $\mathcal{P}_n$ ,  $C_{n+1}$  is distributed uniformly on  $\Delta_n^R$ . Because  $V_{n+1}$  is independent of  $\mathcal{P}_n$ , we conclude that the law of  $Y$  is the one given in Definition 2.1.

Using (46), we see that

$$Y_n = Y'_{J_n} \leq \tilde{Y}_{J_n}. \quad (49)$$

We introduce

$$\begin{cases} \tilde{\varepsilon}_{n+1} := \mathbb{1}_{\tilde{Y}_n(\tilde{C}_{n+1}) \geq \tilde{V}_{n+1}}, \\ \varepsilon_{n+1} := \mathbb{1}_{Y_n(C_{n+1}) \geq V_{n+1}}. \end{cases}$$

Because  $\tilde{f} = a \delta_{B(C_0, r_0)}$ , we can use Proposition 2.5, and we obtain that there exists  $\tilde{\kappa}$  almost surely finite such that

$$\forall n > \tilde{\kappa}, \quad \tilde{\varepsilon}_n = 0. \quad (50)$$

In particular, this implies that there exists  $\kappa$  almost surely finite such that

$$\forall n > \kappa, \quad \tilde{\varepsilon}_{J_n} = 0. \quad (51)$$

Combined with (49), this implies that

$$\forall n > \kappa, \quad \varepsilon_n = 0, \quad (52)$$

and the conclusion follows.  $\square$

## 5.2. The continuous time process is non explosive

We are now going to construct explicitly the process  $(X_t)_{t \geq 0}$  with generator (4) as a continuous time Markov chain, by using  $(Y_n)_{n \geq 0}$  as the embedded Markov chain.

**Definition 5.2.** Consider an i.i.d sequence  $(E_1, E_2, \dots)$  of  $\text{Exp}(1)$  random variables. We define the jump times  $(T_0, T_1, \dots)$  by setting  $T_0 = 0$  and

$$T_n = \frac{E_1}{\lambda(Y_0)} + \dots + \frac{E_n}{\lambda(Y_{n-1})}, \quad n \geq 1, \quad (53)$$

where  $\lambda(f) := |\text{Supp}(f)|^R$  for  $f \in \mathcal{S}_c$ .

We can then define a stochastic process  $(X_t)_{t \geq 0}$  by setting

$$\forall n \geq 0, \forall t \in [T_n, T_{n+1}), \quad X_t = Y_n. \quad (54)$$

We recall that the set  $(\text{Supp}(f))^R$  is the  $R$ -expansion of the support of  $f$ , that is the set of points at a distance less than  $R$  from the support of  $f$ . The quantity  $\lambda(f)$  is its volume, and it is the rate at which the process  $(X_t)_{t \geq 0}$  jumps out of the state  $f$ . This will be verified in the following proposition by checking that we have the correct generator.

**Proposition 5.3.** *The process  $(X_t)_{t \geq 0}$  constructed in (54) is a non-explosive  $\mathcal{S}_c$ -valued continuous time Markov chain. Moreover, its generator is given by (4).*

*Proof.* The first thing to verify is that  $X_t$  is really defined for all nonnegative  $t$ . This is equivalent to saying that

$$\mathbb{P}[T_n \rightarrow \infty \text{ as } n \rightarrow \infty] = 1,$$

that is

$$\mathbb{P}\left[\sum_{n=1}^{\infty} \frac{E_n}{\lambda(Y_{n-1})} = \infty\right] = 1.$$

We show in Proposition 2.5 that  $(\Delta_n)_{n \geq 0}$  converges in a finite number of steps to a bounded set  $B$ . This means that almost surely, there is a random time  $\kappa$  such that for all  $n \geq \kappa$ ,

$$\lambda(Y_n) = |B^R|$$

which implies that

$$\sum_{n=\kappa+1}^{\infty} \frac{E_n}{\lambda(Y_{n-1})} = \frac{\sum_{n=\kappa+1}^{\infty} E_n}{|B^R|} = \infty \text{ a.s.}$$

Hence  $(X_t)_{t \geq 0}$  is a stochastic process defined for all  $t \geq 0$ . The Markov property is obvious, and this shows that  $(X_t)_{t \geq 0}$  is a non-explosive continuous time Markov chain. We can then write the generator of  $(X_t)_{t \geq 0}$  for functions  $G : \mathcal{S}_c \mapsto \mathbb{R}$  as

$$\mathcal{L}G(f) = \int_{(\text{Supp}(f))^R} \int_0^1 G[f + U\delta_{B(c,R)}(\mathbf{1}_{v \leq f(c)} - f)] - G(f) dv dc.$$

If we take  $G = I_n(\cdot, \psi)$  as defined in (3), the generator of  $(X_t)_{t \geq 0}$  takes the form (4).  $\square$

### 5.3. Proof of Theorem 1.3

We have seen in Proposition 5.3 that the process  $(X_t)_{t \geq 0}$  is a non explosive continuous time Markov chain. Therefore, the trajectories of  $(X_t)_{t \geq 0}$  are completely described by its embedded Markov chain  $(Y_n)_{n \geq 0}$ .

In particular, for all  $n \geq 0$ , for all  $t \in [T_n, T_{n+1})$ ,  $\text{Supp}(X_t) = \Delta_n$ . Using the result from Proposition 5.1, and the sequence of times  $(T_n)_{n \geq 0}$  defined in (53), there exists a finite random set  $B \subset \mathbb{R}^d$ , and an almost surely finite random time  $T := T_\kappa$ , such that

$$\forall t > T, \text{Supp}(X_t) = B \quad \text{a.s.} \quad (55)$$

The second point to prove here is the extinction of the population. From Proposition (5.1),

$$\forall n > \kappa, \quad \varepsilon_n = 0. \quad (56)$$

This implies that at every point  $x$ , the frequency  $(X_t(x))_{t \geq T}$  converges geometrically to zero, which concludes the proof.

## 6. Conclusion

Although the SAFV process is constructed in great generality, our study was restricted to the case where  $R$  and  $U$  are constant. In the setting described in [2], these quantities can be made random by adding extra dimensions to the space-time Poisson point process. We then define  $\Pi$  on the space  $[0, \infty) \times \mathbb{R}^d \times [0, 1] \times (0, \infty) \times [0, 1]$ , with intensity  $dt \otimes dc \otimes dv \times \zeta(dr, du)$ , such that

$$\int_{(0, \infty) \times [0, 1]} u r^d \zeta(dr, du) < \infty.$$

Our result holds in the case  $\zeta(dr, du) = \delta_{R, U}(dr, du)$  because the volume of  $\Delta_n$  increases at most linearly with  $n$ . We could imagine extending the same result using practically the same method to the case where

$$\int_{(0, \infty) \times [0, 1]} r^d \zeta(dr, du) < \infty,$$

because the process still jumps at finite rate, and the volume of  $\Delta_n$  is at most of order  $n\mathbb{E}(R)$ , where  $R$  is a realisation of the random radius. The problem comes from the fact that the radii being random makes the construction of the Markov chain more complicated. Morally the result remains true in this case, but the proof becomes significantly more involved.

The situation where

$$\int_{(0, \infty) \times [0, 1]} r^d \zeta(dr, du) = \infty$$

is radically different, because now the process jumps at an infinite rate. The problem is that we do not have a description of the geometry of the process at time  $t > 0$ . The behaviour is not obvious, and it cannot be simulated. For us this remains an open question, which would certainly require different techniques.

## References

- [1] N. H. Barton, A. M. Etheridge, and J. Kelleher. A new model for extinction and recolonization in two dimensions: quantifying phylogeography. *Evolution*, 64:2701–2715, 2010.
- [2] N. H. Barton, A. M. Etheridge, and A. Véber. A new model for evolution in a spatial continuum. *Electronic Journal of Probability*, 15:162–216, 2010.
- [3] N.. Berestycki, A. M. Etheridge, and A. Véber. Large scale behaviour of the spatial Lambda-Fleming-Viot process. <http://arxiv.org/abs/1107.4254>, 2011.
- [4] J. Bertoin and J. F. Le Gall. Stochastic flows associated to coalescent processes. *Probability Theory and Related Fields*, 126:261–288, 2003.
- [5] P. Clifford and A. Sudbury. A model for spatial conflict. *Biometrika*, 60:581–588, 1973.
- [6] P. Donnelly and T.G. Kurtz. Particle representations for measure-valued population models. *Annals of Probability*, 27:166–205, 1999.
- [7] A. M. Etheridge. Drift, draft and structure: some mathematical models of evolution. *Banach Center Publications*, 80:121–144, 2008.
- [8] T.M. Liggett. Stochastic models of interacting systems. *Annals of Probability*, 25:1–29, 1997.